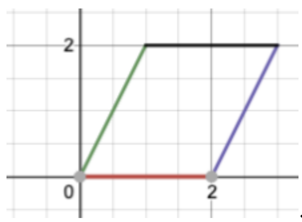
SPRING 2021: MATH 147 QUIZ 5 SOLUTIONS

Each question is worth 5 points. You must justify your answer to receive full credit.

1. Find the volume of the region in $\mathbb{R}^3$ bounded above by the graph of $z = \sin(\pi y/2) + x$ and bounded below by the parallelogram in the xy-plane having vertices (0,0), (2,0), (1, 2), (3,2).

Solution. The region in question is bounded on the left by $x = \frac{y}{2}$ and on the right by $x = \frac{y}{2} + 2$, and is bounded above by $y = 0$ and $y = 2$.



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Therefore,

$$\begin{aligned}
\text{Volume} &= \int_0^2 \int_{\frac{y}{2}}^{\frac{y}{2}+2} \sin(\frac{\pi y}{2}) + x \; dx \; dy \\
&= \int_0^2 (\sin(\frac{\pi y}{2})x + \frac{1}{2}x^2)\Big|_{\frac{y}{2}}^{x=\frac{y}{2}+2} \; dy \\
&= \int_0^2 \{\sin(\frac{\pi y}{2})(\frac{y}{2}+2) + \frac{1}{2}(\frac{y}{2}+2)^2\} - \{\sin(\frac{\pi y}{2})(\frac{y}{2}) + \frac{1}{2}(\frac{y}{2})^2\} \; dy \\
&= \int_0^2 2\sin(\frac{\pi y}{2}) + y + 2 \; dy \\
&= \{-\frac{4}{\pi}\cos(\frac{\pi y}{2}) + \frac{y^2}{2} + 2y\}\Big|_0^2 \\
&= (\frac{4}{\pi} + 2 + 4) - (-\frac{4}{\pi} + 0 + 0) \\
&= \frac{8}{\pi} + 6.
\end{aligned}$$

2. State why it is difficult to evaluate $\int_0^1 \int_y^1 \frac{2y}{x^2+y^2} \; dx \; dy$, and then change the order of integration to evaluate the indicated double integral. (This is the original HW problem.)

Solution. Integrating with respect to x first is more difficult, since a simple u-substitution involving x is not available. However, changing the order of integration, we have:

$$\begin{aligned}
\int_0^1 \int_y^1 \frac{2y}{x^2+y^2}\ dx\ dy &= \int_0^1 \int_0^x \frac{2y}{x^2+y^2}\ dy\ dx \\
&= \int_0^1 \ln(x^2+y^2)\Big|_{y=0}^{y=x}\ dx \quad (\text{via } u-\text{substitution with } u = y^2+x^2) \\
&= \int_0^1 \ln(x^2+x^2) - \ln(x^2+0)\ dx \\
&= \int_0^1 \ln(\frac{2x^2}{x^2})\ dx \\
&= \int_0^1 \ln(2)\ dx \\
&= \ln(2).
\end{aligned}$$

Note: The solution above proceeds in the way that the text by Hartman would expect you to work this problem (# 21 in Section 13.2, in Hartman). However, there is something subtle that is being ignored in this problem, namely, that the integrand $\frac{2y}{x^2+y^2}$ is not defined at (0,0). In fact, $\lim_{(x,y)\to(0,0)} \frac{2y}{x^2+y^2}$ does not exist. The given integral is an example of an improper double integral that converges, something we will talk about in class, that is not covered in Hartman's book.